SUMMABILITY AND THE CLOSED GRAPH THEOREM

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Abstract. This note provides a short illustration of the Silverman-Toeplitz theorem from the functional analysis point of view.

1. Introduction

According to [2, p. 148], an infinite-dimensional complex matrix $T = (t_{mn})_{m,n=1}^{\infty}$ is said to be regular if it satisfies the following conditions:

(i) There exists a constant $C = C(T) > 0$ such that $\displaystyle\sum_{n=1}^{\infty} |t_{mn}| \le C$ for all $m \ge 1$;

(ii) For every $n \ge 1$, we have $\displaystyle\lim_{m\to\infty} t_{mn} = 0$;

(iii) For every $n \ge 1$, we have $\displaystyle\lim_{m\to\infty} \sum_{n=1}^{\infty} t_{mn} = 1$.

It is shown [2, Theorem 5.5] that regular matrices preserve limits. More precisely, if $T = (t_{mn})$ is regular and $a_n \to a \in \mathbb{C}$ as $n \to \infty$, then

$$b_m = \sum_{n=1}^{\infty} t_{mn} a_n \tag{1.1}$$

is well-defined for each $m \ge 1$ and $b_m \to a$ as $m \to \infty$. The converse is also true [2, Exercise 5.2.1.3, p. 157]; that is, if T preserves limits in the sense that given any sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ converging to $a \in \mathbb{C}$, the sequence $\{b_m\}_{m=1}^{\infty}$ given by (1.1) is well-defined for each $m \ge 1$ and $b_m \to a$ as $m \to \infty$, then T must be regular. This result together with [2, Theorem 5.5] is now known as the Silverman-Toeplitz theorem. It is clear that if T preserves limits, then it satisfies the conditions (ii) and (iii). Now we show, using tools from functional analysis, that if T preserves limits, then it also satisfies (i).

2. Normed Vector Spaces and Linear Operators

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let X be an $\mathbb{F}$-vector space. A norm on X is a function $\|\cdot\| \colon X \to \mathbb{R}_{\ge 0}$ satisfying the following properties:

(a) Positive-definiteness: for any $x \in X$, $\|x\| \ge 0$ with equality if and only if $x = 0$;

(b) Absolute homogeneity: for any $x \in X$ and $c \in \mathbb{F}$, we have $\|cx\| = |c|\|x\|$;

(c) Triangle inequality: for any $x, y \in X$, we have $\|x + y\| \le \|x\| + \|y\|$.

A normed vector space $(X, \|\cdot\|)$ is simply a vector space X equipped with a norm $\|\cdot\|$. The norm $\|\cdot\|$ induces a topology $\mathcal{T}$ on X. We say that $(X, \|\cdot\|)$ is a Banach space if X is complete with respect to $\mathcal{T}$. The finite-dimensional complex vector space $(\mathbb{C}^n, \|\cdot\|_2)$ provides the simplest example of a complex Banach space, where

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

for all $x \in \mathbb{C}^n$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces with induced topology $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. A linear operator $T\colon X \to Y$ is an $\mathbb{F}$-linear map from X to Y. The set of all linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$, and we shall write $\mathcal{L}(X) := \mathcal{L}(X, X)$ for simplicity. We say that T is continuous if $T\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous function. For a given linear operator $T\colon X \to Y$, we define the norm of T by

$$\|T\| := \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

It is clear that

$$\|T\| = \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \|T(x)\|_Y = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|T(x)\|_Y.$$

We say that T is bounded if $\|T\| < \infty$. The set of all bounded linear operators from X to Y is denoted by $\mathcal{B}(X, Y)$, and similarly we shall write $\mathcal{B}(X) := \mathcal{B}(X, X)$. It can be shown [1, Proposition 2.1, Chapter III] that T is bounded if and only if T is continuous. One of the most important results concerning bounded linear operators is the following known as the closed graph theorem [1, Theorem 12.6, Chapter III].

Theorem 2.1 (Closed Graph Theorem)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. A linear operator $T\colon X \to Y$ is bounded if and only if the graph of T,*

$$\mathrm{Gr}(T) := \{(x, T(x)) \in X \times Y : x \in X\},$$

is closed in $X \times Y$.

Equivalently, a linear operator $T\colon X \to Y$ between Banach spaces X and Y is bounded if and only if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $x_n \to x \in X$ and $T(x_n) \to y \in Y$ as $n \to \infty$, one has $y = T(x)$. We shall use this equivalent formulation of Theorem 2.1 to show that a matrix that preserves limits must satisfy (i).

3. ℓ^∞ and Its Subspaces

It is well known that the space $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space, where

$$\|x\|_\infty := \sup_{n \ge 1} |x_n|$$

for any infinite sequence $x = (x_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ and

$$\ell^\infty := \left\{(x_n)_{n=1}^{\infty} \subseteq \mathbb{C} \colon \|x\|_\infty < \infty\right\}.$$

Consider the subspace $(\ell_c^\infty, \|\cdot\|_\infty)$, where ℓ_c^∞ consists of all the convergent sequences in ℓ^∞. It is not hard to see that ℓ_c^∞ is closed in ℓ^∞ and hence a Banach space. Indeed, suppose that $\{x^k\}_{k=1}^{\infty} \subseteq \ell_c^\infty$ converges to $x \in \ell^\infty$. Let $\epsilon > 0$ be arbitrary. Then there exists $K \ge 1$

such that $|x_n^K - x_n| < \epsilon/3$ for all $n \geq 1$. Since $x^K \in \ell_c^\infty$, there exists $N \geq 1$ such that $|x_m^K - x_n^K| < \epsilon/3$ for all $m, n \geq N$. It follows that

$$|x_m - x_n| \leq |x_m - x_m^K| + |x_m^K - x_n^K| + |x_n^K - x_n| < \epsilon$$

for all $m, n \geq N$. Thus $\{x_n\}_{n=1}^\infty$ is convergent, which implies that $x \in \ell_c^\infty$. This proves that ℓ_c^∞ is closed in ℓ^∞. In fact, if $x_n^k \to a_k$ and $x_n \to a$ as $n \to \infty$, then we see that $a_k \to a$ as $k \to \infty$ by considering the inequality

$$|a_k - a| \leq |a_k - x_n^k| + |x_n^k - x_n| + |x_n - a|.$$

Consider an arbitrary linear operator $T \colon \ell_c^\infty \to \ell_c^\infty$ on $(\ell_c^\infty, \|\cdot\|_\infty)$ with matrix representation $T = (t_{mn})$.[1] Note that

$$\|T\| = \sup_{\substack{x \in \ell^\infty \\ \|x\|_\infty \leq 1}} \|Tx\|_\infty = \sup_{\substack{x \in \ell^\infty \\ \|x\|_\infty \leq 1}} \|(b_m(x))_{m=1}^\infty\|_\infty = \sup_{\substack{x \in \ell^\infty \\ \|x\|_\infty \leq 1}} \sup_{m \geq 1} |b_m(x)|,$$

where

$$b_m(x) := \sum_{n=1}^\infty t_{mn} x_n.$$

Clearly, we have $\|T\| \leq C(T)$, where

$$C(T) := \sup_{m \geq 1} \sum_{n=1}^\infty |t_{mn}| \in [0, +\infty].$$

On the other hand, suppose that $m \geq 1$ and $N \geq 1$ are positive integers. Define $(x_n)_{n=1}^\infty$ by $x_n = \operatorname{sgn}(t_{mn})$ if $n \leq N$ and $x_n = 0$ otherwise, where $\operatorname{sgn}(z) = 0$ if $z = 0$ and $\operatorname{sgn}(z) = |z|/z$ if $z \neq 0$. Then $x_n \to 0$ as $n \to \infty$ and $\|(x_n)_{n=1}^\infty\|_\infty \leq 1$. It follows that

$$\|T\| \geq |b_m(x)| = \sum_{n=1}^N |t_{mn}|.$$

Since $N \geq 1$ and $m \geq 1$ are arbitrary, we conclude that $\|T\| \geq C(T)$. Hence $\|T\| = C(T)$.

Let $T = (t_{mn})$ be a matrix that preserves limits. Then $T \colon \ell_c^\infty \to \ell_c^\infty$ is a linear operator on $(\ell_c^\infty, \|\cdot\|_\infty)$. We show that

$$\sum_{n=1}^\infty |t_{mn}| < \infty$$

for every $m \geq 1$. Suppose that this is false for some $m \geq 1$. Then there exists a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that $n_1 = 1$ and

$$\sum_{n=n_k}^{n_{k+1}-1} |t_{mn}| \geq k \tag{3.1}$$

[1] It is important to note that not every linear operator on $(\ell_c^\infty, \|\cdot\|_\infty)$ has a matrix representation, though bounded ones do have representing matrices with respect to a Schauder basis for ℓ_c^∞, say $\{e_r\}_{r=0}^\infty$, where $e_0 := (1, 1, \ldots)$ and $e_r = (x_n)_{n=1}^\infty \in \ell^\infty$ with $x_r = 1$ and $x_n = 0$ for all $n \neq r$ when $r \geq 1$, so that every $(x_r)_{r=1}^\infty \in \ell_c^\infty$ with $x_r \to x \in \mathbb{C}$ as $r \to \infty$ can be (uniquely) written as

$$(x_r)_{r=1}^\infty = x e_0 + \sum_{r=1}^\infty (x_r - x) e_r.$$

for all $k \geq 1$. Taking $(x_n)_{n=1}^{\infty}$ with $x_n = \operatorname{sgn}(t_{mn})/k$ for all $n \in [n_k, n_{k+1})$ and observing that $x_n \to 0$ as $n \to \infty$, we have

$$b_m(x) = \sum_{k=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} \frac{|t_{mn}|}{k} < \infty.$$

But (3.1) implies that $b_m(x) = \infty$, a contradiction.

Now we show that $T \in \mathcal{B}(\ell_c^{\infty})$. In view of Theorem 2.1, we need only to prove that for any $\{x^k\}_{k=1}^{\infty} \subseteq \ell_c^{\infty}$ such that $x^k \to x \in \ell_c^{\infty}$ and $Tx^k \to y \in \ell_c^{\infty}$ as $k \to \infty$, we have $y = Tx$. Suppose that $x_n^k \to a_k$, $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Then $a_k \to a$ as $k \to \infty$ and $(Tx)_m \to a$ as $m \to \infty$. Let $\epsilon > 0$. Since $Tx^k \to y$ as $k \to \infty$, there exists $K \geq 1$ such that

$$\left| \sum_{n=1}^{\infty} t_{mn} x_n^k - y_m \right| < \epsilon$$

for all $k \geq K$ and all $m \geq 1$. Letting $m \to \infty$ we obtain $|a_k - b| \leq \epsilon$ for all $k \geq K$. Since $\epsilon > 0$ is arbitrary, we see that $a_k \to b$ as $k \to \infty$. Hence $a = b$. This implies that for any $\epsilon > 0$, there exists $M \geq 1$ such that $|(Tx)_m - y_m| < \epsilon$ for all $m > M$. Put

$$C_M := \max_{1 \leq m \leq M} \sum_{n=1}^{\infty} |t_{mn}| < \infty.$$

Since $x^k \to x$ as $k \to \infty$, we have

$$|(Tx)_m - y_m| = \lim_{k \to \infty} \left| \sum_{n=1}^{\infty} t_{mn}(x_n - x_n^k) \right| \leq C_M \cdot \lim_{k \to \infty} \|x - x^k\|_{\infty} = 0$$

for all $1 \leq m \leq M$. Hence $\|Tx - y\|_{\infty} \leq \epsilon$. We conclude that $y = Tx$.

In general, we may define regular operators on ℓ_c^{∞}. Denote by ℓ_0^{∞} the closed subspace of ℓ_c^{∞} consisting of all the sequences $(x_n)_{n=1}^{\infty}$ such that $x_n \to 0$ as $n \to \infty$. For each $z \in \mathbb{C}$, let

$$\ell_0^{\infty} + z := \{(x_n + z)_{n=1}^{\infty} \colon (x_n)_{n=1}^{\infty} \in \ell_0^{\infty}\} \subseteq \ell_c^{\infty}.$$

Then ℓ_c^{∞} is the disjoint union of $\ell_0^{\infty} + z$ over $z \in \mathbb{C}$. We say that $T \in \mathcal{L}(\ell_c^{\infty})$ is weakly regular if $T \in \mathcal{B}(\ell_c^{\infty})$ and $T(e_r) \in \ell_0^{\infty}$ for all $r \geq 1$. Clearly, if T is weakly regular, then $T|_{\ell_0^{\infty}} \in \mathcal{B}(\ell_0^{\infty})$, since $\{e_r\}_{r=1}^{\infty}$ is a Schauder basis for ℓ_0^{∞}. However, the converse may not hold. We say that T is regular if T is weakly regular such that $T(e_0) \in \ell_0^{\infty} + 1$. On the other hand, we say that $T \in \mathcal{L}(\ell_c^{\infty})$ preserves limits if $T|_{\ell_0^{\infty}} \in \mathcal{B}(\ell_0^{\infty})$ and $T(\ell_0^{\infty} + z) \subseteq \ell_0^{\infty} + z$ for all $z \in \mathbb{C}$. Then one can show that $T \in \mathcal{L}(\ell_c^{\infty})$ is regular if and only if T preserves limits.

References

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[2] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Stud. Adv. Math., vol. 97, Cambridge Univ. Press, Cambridge, 2006.

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